View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by Elsevier - Publisher Connector

Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Journal of Computational and Applied Mathematics 187 (2006) 72–86
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

Group solution for unsteady free-convection flow from a vertical moving plate subjected to constant heat flux

M. Kassem*

Department of Engineering Mathematics and Physics, Faculty of Engineering, Zagazig University, Zagazig, Egypt

Received 30 May 2004; received in revised form 11 November 2004

Abstract

The problem of heat and mass transfer in an unsteady free-convection flow over a continuous moving vertical sheet in an ambient fluid is investigated for constant heat flux using the group theoretical method. The nonlinear coupled partial differential equation governing the flow and the boundary conditions are transformed to a system of ordinary differential equations with appropriate boundary conditions. The obtained ordinary differential equations are solved numerically using the shooting method. The effect of Prandlt number on the velocity and temperature of the boundary-layer is plotted in curves. A comparison with previous work is presented.
© 2005 Elsevier B.V. All rights reserved.

MSC: 22E05; 35Q53; 54H15

Keywords: Free convective flow; Group theoretic method; Thermal boundary layer; Prandtl number

1. Introduction

The problem of free convection about a continuously moving flat plate has many practical applications in manufacturing process. Many attempts were made to find analytical and numerical solutions of the problem using different mathematical appropriate methods. In 1969, Heinisch et al. [4] using an integral technique, obtained a system of partial differential equations in two independent variables. This resultant system was again reduced to a system of ordinary differential equations by two separate methods. The first was the method of integral relations; the second was an explicit finite-difference scheme. Approximate

* Tel.: +210 30 23 860; fax: +202 760 37 35.
E-mail address: mkassem@thewayout.net.

0377-0427/$ - see front matter © 2005 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2005.03.037

temperature and velocities were obtained by both methods. More recently in 1987, Williams et al. [10] obtained semi-similar solutions for the unsteady free-convective boundary layer flow on a vertical plate. An implicit finite-difference scheme was used to solve the resultant ordinary differential equations. The method was tested for a number of possible surface-temperature variation with time and position. In 1996, Kumari et al. [5] investigated free-convective flow on a vertical plate subjected to constant temperature and constant heat flux. The set of ordinary differential equations resulting from a semi-similarity transformation of the problem was solved numerically using two finite difference schemes (Keller box and Nakamura method). Three cases of surface velocity distributions representing accelerating, decelerating and decaying flow were tested, as well as the variation of Nusselt number for each case. In 2003, Pantangi et al. [9] analyze the free-convective flow on a vertical plate subjected to a constant heat flux, for a non-Newtonian fluid using a semi-similar method. The resultant ordinary differential equations were solved using a quasilinearization scheme based on a truncated Taylor series. The temperature and velocity distributions inside the boundary layer were plotted for various flow indexes "n" and different Prandtl numbers.

Finally, the group method developed in 1968 by Moran and Gaggioli [7,8] was used by Boutros et al. in 1990 [3] for the solution of a steady-state free convective flow on a flat plate. Two cases of temperature varying with position were tested for different Prandtl numbers. In 1990, Abd-el-Malek and Badran [1] applied the group method for the analysis of unsteady free convective flow on a flat plate for the case where the velocity of flow next to the wall equal to zero. Several forms of surface temperature varying with time and position were derived and the analytical and numerical results compared with previous work. Recently Abd-el-Malek et al. in 2004 [2] investigated free-convective flow about a flat plate for constant wall temperature using the group method. The temperature distribution inside the boundary layer was derived analytically while the velocity inside the layer was evaluated numerically. Different Prandlt numbers were tested.

Here we analyze unsteady free-convective flow over a continuous moving vertical plate subjected to a constant heat flux and with the velocity next to the wall different from zero. The application of a two-parameter group to the governing partial differential equations and boundary conditions reduces the problem to a system of ordinary differential equations with related boundary conditions. This system of equations is solved numerically using the shooting method.

2. Mathematical formulation

The unsteady laminar flow of an incompressible fluid caused by a continuous moving sheet placed in a fluid at rest is considered. The flat sheet illustrated in Fig. 1 issues from a thin slit at $x = y = 0$ and is subsequently stretched vertically. The positive x-coordinate is measured along the direction of the moving sheet with the slot as the origin and the direction "y" is measured normal to the sheet. The flat sheet is subjected to a constant heat flux q_{w}, giving rise to a buoyancy force, while the ambient fluid is kept at a constant temperature T_∞.

The boundary layer equations governing the free convection flow over the moving sheet are expressed in the form

$$\frac{\partial u^*}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

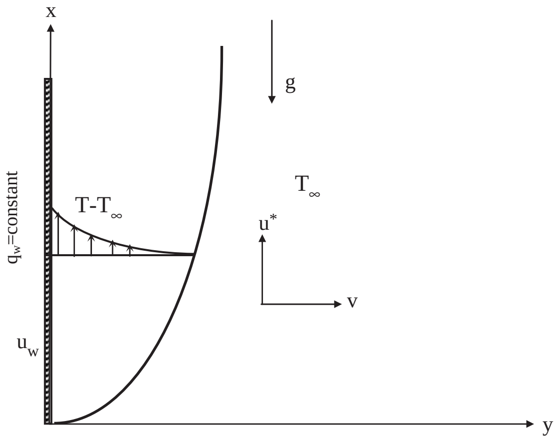

Fig. 1. Illustration of unsteady laminar flow over, a moving vertical plate.

$$\frac{\partial u^*}{\partial t} + u^* \frac{\partial u^*}{\partial x} + v \frac{\partial u^*}{\partial y} = \nu \frac{\partial^2 u^*}{\partial y^2} + g\beta(T - T_\infty), \tag{2}$$

$$\frac{\partial T}{\partial t} + u^* \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{\nu}{Pr} \frac{\partial^2 T}{\partial y^2}, \tag{3}$$

with initial and boundary conditions:

$$u^*(x, y, 0) = u_0^*(x, y), \quad v(x, y, 0) = v_0(x, y), \quad T(x, y, 0) = T_0(x, y),$$
$$u^*(x, y)' = u_w(x, t), \quad v(x, 0, t) = 0, \quad \frac{\partial T(x, 0, t)}{\partial y} = \frac{-q_w}{k}, \tag{4.1}$$

$$u^*(x, y, t) = 0, \quad T(x, y, t) = T_\infty \quad \text{for } y \to +\infty, \tag{4.2}$$

where u^* and v are the velocity components along the x and y directions, respectively; T is the temperature; g is the acceleration due to gravity; Pr is the Prandtl number $Pr = \nu/\alpha$, where "ν" kinematics viscosity and "α" is the thermal diffusivity; β is the coefficient of thermal expansion; k is the thermal diffusivity coefficient. u_0, v_0 and T_0 are the initial velocity x, y components and the initial temperature, respectively; u_w is the velocity of the fluid at the wall.

At this point, we introduce the nondimensional velocity and temperature

$$u(x, y, t) = \frac{u^*(x, y, t)}{u_w(x, t)} \quad \text{and} \quad \theta(x, y, t) = \frac{T(x, y, t) - T_\infty}{q_w x} k\sqrt{Re_x},$$

where Re_x is the local Reynolds number.

Reducing (1–3) to:

$$u \frac{\partial u_w}{\partial x} + u_w \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{5}$$

$$u\,\frac{\partial u_w}{\partial t}+u_w\,\frac{\partial u}{\partial t}+u_w^2u\,\frac{\partial u}{\partial x}+u^2u_w\,\frac{\partial u_w}{\partial x}+vu_w\,\frac{\partial u}{\partial y}=\nu u_w\,\frac{\partial^2 u}{\partial y^2}+\frac{g\beta q_w}{k\sqrt{Re_x}}\,x\theta, \tag{6}$$

$$x\,\frac{\partial \theta}{\partial t}+u_wu\left(\frac{\partial \theta}{\partial x}x+\theta\right)+v\,\frac{\partial \theta}{\partial y}x=\frac{\nu}{Pr}\,x\,\frac{\partial^2\theta}{\partial y^2}, \tag{7}$$

with initial and boundary conditions

$$\begin{aligned}
&u(x,y,0)=u_0(x,y),\quad v(x,y,0)=v_0(x,y),\,\theta(x,y,0)=\theta_0(x,y),\\
&u(x,0,t)=1,\,v(x,0,t)=0,\quad \frac{\partial\theta(x,0,t)}{\partial y}=-\frac{\sqrt{Re_x}}{x},\\
&u(x,y,t)=0,\quad \theta(x,y,t)=0\quad \text{for } y\to+\infty.
\end{aligned} \tag{8}$$

3. Group formulation of the problem

Group G, a class of two parameters (a_1,a_2) is used to reduce the flow Eqs. (6–8) to a system of ordinary equations in one variable $\eta(x,y,t)$. This group formulates as

$$G:\ \bar S=C^S(a_1,a_2)S+K^S(a_1,a_2), \tag{9}$$

where S stands for $(x,y,t;u,u_w,v$ and $\theta)$. C^s and K^s are real valued and at least differentiable in their real arguments (a_1,a_2).

First- and second-order partial derivatives of the variables $(u,u_w,v$ and $\theta)$ are obtained from G via chain rule operations:

$$\left.\begin{aligned}\bar S_{\bar i}&=(C^s/C^i)S_i,\\ \bar S_{\bar i\bar j}&=(C^s/C^iC^j)S_{ij},\end{aligned}\right\}\ i,j=x,y,t, \tag{10}$$

where the subscripts 'i', 'j' stands for differentiation with respect to x,y,t.

3.1. Transformation

Eq. (5) is transformed to

$$\bar u\,\frac{\partial \bar u_w}{\partial \bar x}+\bar u_w\,\frac{\partial \bar u}{\partial \bar x}+\frac{\partial \bar v}{\partial \bar y}=H_1(a_1,a_2)\left[u\,\frac{\partial u_w}{\partial x}+u_w\,\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right], \tag{11}$$

where $H_1(a_1,a_2)$ is an arbitrary function of the two-group parameter (a_1,a_2)

From the group definitions given in (9) and (10), (11) reduces to

$$\begin{aligned}
&[C^uC^{u_w}/C^x]u\,\frac{\partial u_w}{\partial x}+[C^uC^{u_w}/C^x]u_w\,\frac{\partial u}{\partial x}+[C^v/C^y]\,\frac{\partial v}{\partial y}+R_1\\
&\quad=H_1(a_1,a_2)\left[u\,\frac{\partial u_w}{\partial x}+u_w\,\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right],
\end{aligned} \tag{12}$$

where

$$R_1=[C^{u_w}K^u/C^x]\,\frac{\partial u_w}{\partial x}+[C^uK^{u_w}/C^x]\,\frac{\partial u}{\partial x}.$$

For invariant transformation R_1 is equated to zero. This is satisfied by setting

$$K^u = K^{u_\mathrm{w}} = 0. \tag{13}$$

Comparing the coefficients in both sides of (11) and with $H_1(a_1, a_2)$ equated to one for absolute invariance, we obtain

$$\left[\frac{C^u C^{u_\mathrm{w}}}{C^x}\right] = \left[\frac{C^v}{C^y}\right] = 1. \tag{14}$$

Similarly Eq. (6) is invariantly transformed to

$$\begin{aligned}
&[C^u C^{u_\mathrm{w}}/C^t]u_\mathrm{w}\,\frac{\partial u}{\partial t}[C^u C^{u_\mathrm{w}}/C^t]\,u\,\frac{\partial u_\mathrm{w}}{\partial t} + [(C^u C^{u_\mathrm{w}})^2/C^x]\left(\frac{\partial u}{\partial x} + \frac{\partial u_\mathrm{w}}{\partial x}\right) \\
&\quad + [C^v C^u C^{u_\mathrm{w}}/C^y]vu_\mathrm{w}\,\frac{\partial u}{\partial y} - v(C^{u_\mathrm{w}} C^u/(C_y)^2)\frac{\partial^2 u}{\partial y^2} - \frac{g\beta q_\mathrm{w}}{k\sqrt{Re_x}}(C^\theta C^x)\theta x + R_2 \\
&\quad = H_2(a_1, a_2)\begin{bmatrix} u_\mathrm{w}\,\dfrac{\partial u}{\partial t} + u\,\dfrac{\partial u_\mathrm{w}}{\partial t} + uu_\mathrm{w}^2\,\dfrac{\partial u}{\partial x} + u^2 u_\mathrm{w}\,\dfrac{\partial u_\mathrm{w}}{\partial x} \\ +vu_\mathrm{w}\,\dfrac{\partial u}{\partial y} - vu_\mathrm{w}\,\dfrac{\partial^2 u}{\partial y^2} - \dfrac{g\beta q_\mathrm{w}\theta}{k\sqrt{Re_x}}x \end{bmatrix},
\end{aligned} \tag{15}$$

where R_2 is expressed as

$$\begin{aligned}
R_2 = {} & \left[K^{u_\mathrm{w}}\,\frac{C^u}{C^t}\right]\frac{\partial u}{\partial t} + \left[K^u\,\frac{C^{u_\mathrm{w}}}{C^t}\right]\frac{\partial u_\mathrm{w}}{\partial t} \\
& + \{K^u(C^{u_\mathrm{w}}u_\mathrm{w} + K^{u_\mathrm{w}})^2 + [(K^{u_\mathrm{w}})^2 + 2K^{u_\mathrm{w}}C^{u_\mathrm{w}}u_\mathrm{w}](C^u u + K^u)\}\frac{C^u}{C^x}\,\frac{\partial u}{\partial x} \\
& + \{K^{u_\mathrm{w}}(C^u u + K^u)^2 + [(K^u)^2 + 2K^u C^u u](C^{u_\mathrm{w}}u_\mathrm{w} + K^{u_\mathrm{w}})\}\frac{C^{u_\mathrm{w}}}{C^x}\,\frac{\partial u_\mathrm{w}}{\partial x} \\
& + \{K^v(C^{u_\mathrm{w}}u_\mathrm{w} + K^{u_\mathrm{w}}) + K^{u_\mathrm{w}}(C^v v + K^v)\}\,\frac{C^u}{C^y}\,\frac{\partial u}{\partial y} - v(K^{u_\mathrm{w}}C^u/(C_y)^2)\,\frac{\partial^2 u}{\partial y^2} \\
& - \frac{g\beta q_\mathrm{w}}{k\sqrt{Re_x}}\{K^x(C^\theta\theta + K^\theta) + K^\theta(C^x x + K^x)\}
\end{aligned} \tag{16}$$

$R_2 = 0$ implies that

$$K^v = K^\theta = K^y = 0 \tag{17}$$

and

$$\left[\frac{C^u C^{u_\mathrm{w}}}{C^t}\right] = \left[\frac{(C^u C^{u_\mathrm{w}})^2}{C^x}\right] = \left[\frac{C^u C^v C^{u_\mathrm{w}}}{C^y}\right] = \left[C^{u_\mathrm{w}}\left(\frac{C^u}{C^y}\right)^2\right] = C^\theta C^y = 1, \tag{18}$$

where $H_2(a_1, a_2)=1$ for absolute invariance. Following similar steps (7) is invariantly transformed giving

$$K^x = 0$$

and

$$\left[\frac{C^\theta C^x}{C^t}\right] = [C^u C^{u_\mathrm{w}} C^\theta] = C^v C^x \,\frac{C^\theta}{C^y} = \left[C^x\left(\frac{C^\theta}{C^y}\right)^2\right] = 1. \tag{19}$$

Moreover, the invariance of the initial and boundary conditions in (8) implies that

$$K^t = 0, \tag{20}$$

$$C^u = 1. \tag{21}$$

Invoking (20–21), Eqs. (14), (18) and (19) reduces to

$$C^t = (C^y)^2, \quad C^x = C^y, \quad C^{u_\omega} = C^v = \frac{1}{C^y} \tag{22}$$

which summarize in a group G of the form

$$G \begin{cases} G_s \begin{cases} \bar{x} = C^y x, \\ \bar{y} = C^y y, \\ \bar{t} = (C^y)^2 t, \end{cases} \\ \begin{cases} \bar{u} = u, \\ \bar{u}_\mathrm{w} = \dfrac{1}{C^y}\, u_\mathrm{w}, \\ \bar{v} = \dfrac{1}{C^y}\, v, \\ \bar{\theta} = \theta. \end{cases} \end{cases} \tag{23}$$

4. Group transformation of the boundary layer flow equations

The system of partial differential equations describing the boundary layer flow is transformed through the application of Morgan [6] basic theorem. This theorem states that the two-parameter group G is absolutely invariant if it satisfies the following two first-order linear differential equations:

$$\sum_{i=1}^{7} (\alpha_i S_i + \alpha_{i+1}) \frac{\partial g_i}{\partial S_i} = 0,$$

$$\sum_{i=1}^{7} (\beta_i S_i + \beta_{i+1}) \frac{\partial g_i}{\partial S_i} = 0, \tag{24}$$

where S_i stands for the variables: $x, y, t; u, v, u_{\mathrm{w}}, \theta$ and g_i is the group regulating each variable and the subscript $i = 1, 2, \ldots, 7$.

$$\alpha_1 = \frac{\partial C^x}{\partial a_1}(a_1^0, a_2^0), \quad \alpha_2 = \frac{\partial K^x}{\partial a_1}(a_1^0, a_2^0),$$

$$\beta_1 = \frac{\partial C^x}{\partial a_2}(a_1^0, a_2^0), \quad \beta_2 = \frac{\partial K^x}{\partial a_2}(a_1^0, a_2^0) \ldots \, .$$

where (a_1^0, a_2^0) are the identity elements of the group.

4.1. Transformation of the independent variables

The independent variables are reduced to a single variable $\eta(x, y, t)$ through the application of (24)

$$\begin{aligned}
&(\alpha_1 x)\,\frac{\partial \eta}{\partial x} + \alpha_3 y\,\frac{\partial \eta}{\partial y} + \alpha_5 t\,\frac{\partial \eta}{\partial t} = 0,\\
&(\beta_1 x)\,\frac{\partial \eta}{\partial x} + \beta_3 y\,\frac{\partial \eta}{\partial y} + \beta_5 t\,\frac{\partial \eta}{\partial t} = 0,
\end{aligned} \tag{25}$$

where $\alpha_2 = \beta_2 = 0$ as $K^x = 0$, $\alpha_4 = \beta_4 = 0$ as $K^y = 0$, $\alpha_6 = \beta_6 = 0$ as $K^t = 0$.

A successive elimination of $y\partial\eta/\partial y$ and $\partial\eta/\partial x$ in (25) gives

$$\begin{aligned}
&(\lambda_{13} x)\,\frac{\partial \eta}{\partial x} + \lambda_{53} t\,\frac{\partial \eta}{\partial t} = 0,\\
&\lambda_{13} y\,\frac{\partial \eta}{\partial y} + \lambda_{15} t\,\frac{\partial \eta}{\partial t} = 0,
\end{aligned} \tag{26}$$

where $\lambda_{ij} = \alpha_i \beta_j - \alpha_j \beta_i \quad (i, j = 1, 2, 3, 4, 5)$.

The first equation of (26) has the general solution

$$\eta = f(y, \varepsilon(x, t)), \tag{27}$$

where $\varepsilon(x, t) = xt^a$, $a = \lambda_{13}/\lambda_{53}$ and f is an arbitrary function, that might be equated to one without a loss of generality. η is function of y to satisfy the second equation in (26).

Solving the second equation in (26) through the chain rule we obtain

$$\eta = y\varepsilon^{\gamma}(x, t), \quad \gamma = -\frac{\lambda_{15}}{\lambda_{53}}$$

or

$$\eta = y\pi(x, t), \tag{28}$$

where $\pi(x, t) = (xt^a)^{\gamma}$. a, γ and $\pi(x, t)$ will be determined later.

4.2. Transformation of dependent variables

The dependent variables $u(x, y, t)$ and $\theta(x, y, t)$ are transformed directly from the group transformation described in (23)

$$u(x, y, t) \Longrightarrow u(\eta) \quad \text{and} \quad \theta(x, y, t) \Longrightarrow \theta(\eta), \tag{29}$$

while the transformation of $u_{\mathrm{w}}(x, t)$ and $v(x, y, t)$ is obtained through the application of Morgan theorem (24)

$$\begin{aligned}
&(\alpha_1 x)\frac{\partial g_3(x, y, t; u_{\mathrm{w}})}{\partial x} + (\alpha_5 t)\frac{\partial g_3(x, y, t; u_{\mathrm{w}})}{\partial t} + (\alpha_{11} u_{\mathrm{w}} + \alpha_{12})\frac{\partial g_3(x, y, t; u_{\mathrm{w}})}{\partial u_{\mathrm{w}}} = 0, \\
&(\beta_1 x)\frac{\partial g_3(x, y, t; u_{\mathrm{w}})}{\partial x} + (\beta_5 t)\frac{\partial g_3(x, y, t; u_{\mathrm{w}})}{\partial t} + (\beta_{11} u_{\mathrm{w}} + \beta_{12})\frac{\partial g_3(x, y, t; u_{\mathrm{w}})}{\partial u_{\mathrm{w}}} = 0.
\end{aligned} \tag{30}$$

Eliminating $x\partial g_3/\partial x$, $t\partial g_3/\partial t$ successively, we get

$$g_3(x, t; u_{\mathrm{w}}) = \phi_1(u_{\mathrm{w}}/\omega(x, t)) = E(\eta). \tag{31}$$

In a similar manner, $g_4(x, t; v)$ is obtained

$$g_4(x, t; v) = \phi_2(v/\Gamma(x, t)) = F(\eta), \tag{32}$$

where $\omega(x, t)$ and $\Gamma(x, t)E(\eta)$, $F(\eta)$ are functions to be determined. Without loss of generality, the ϕ's in (30) and (31) are selected to be identity functions, hence the functions $u_{\mathrm{w}}(x, t)$ and $v(x, y, t)$ are expressed in terms $E(\eta)$ and $F(\eta)$

$$u_{\mathrm{w}}(x, t) = \omega(x, t)E(\eta), \tag{33}$$

$$v(x, y, t) = \Gamma(x, t)F(\eta). \tag{34}$$

Since $\omega(x, t)$ and $u_{\mathrm{w}}(x, t)$ are independent of y, whereas η depends on y, it follows that $E(\eta)$ must be equal to a constant, say one, thus (33) becomes

$$u_{\mathrm{w}}(x, t) = \omega(x, t). \tag{35}$$

5. Reduction of the flow equations to a system of ordinary differential equations

The governing Eqs. (5)–(7) are reduced to a set of ordinary differential equations in $F(\eta)$, $\theta(\eta)$, $u(\eta)$. This is realized by substituting (28), (29) and (34), (35) into the first (5), and dividing by $\pi\Gamma$

$$\frac{\mathrm{d}F}{\mathrm{d}\eta} + \frac{\mathrm{d}u}{\mathrm{d}\eta}\eta\left(\frac{\omega}{\Gamma\pi^2}\frac{\partial\pi}{\partial x}\right) + u\left(\frac{1}{\pi\Gamma}\frac{\partial\omega}{\partial x}\right) = 0, \tag{36}$$

similarly substituting (28), (29) and (34), (35) in (6) and dividing by $-v\omega\pi^2$, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}^2 u}{d\eta^2} - \frac{\mathrm{d}u}{\mathrm{d}\eta}\left[\eta\left(\frac{1}{v\pi^3}\frac{\partial\pi}{\partial t} + u\frac{\omega}{v\pi^3}\frac{\partial\pi}{\partial x}\right) + \frac{\Gamma}{v\pi}F\right] - u\left(\frac{1}{v\omega\pi^2}\frac{\partial\omega}{\partial t}\right) - u^2\left(\frac{\partial\omega}{\partial x}\frac{1}{v\pi^2}\right) \\
&\quad + \theta\frac{g\beta q_{\mathrm{w}}}{k\sqrt{Re_x}}\frac{x}{v\omega\pi^2} = 0.
\end{aligned} \tag{37}$$

In a similar way (7) reduces after dividing by $-(v/Pr)x\pi^2$ to

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}\eta^2} - \mathrm{Pr}\left[\frac{\mathrm{d}\theta}{\mathrm{d}\eta}\left\{\eta\left(\frac{1}{v\pi^3}\frac{\partial\pi}{\partial t} + u\,\frac{\omega}{v\pi^3}\frac{\partial\pi}{\partial x}\right) + F\,\frac{\Gamma}{v\pi}\right\} - \theta u\,\frac{\omega}{vx\pi^2}\right] = 0. \tag{38}$$

For (36)–(38) to reduce to a system of equations in a single variable η, it is necessary that the coefficients of the functions $F(\eta), u(\eta), \theta(\eta)$ and their derivatives, be constants or functions of η only. These coefficients are

$$C_1 = \frac{\omega}{\Gamma\pi^2}\frac{\partial\pi}{\partial x}, \tag{39}$$

$$C_2 = \frac{1}{\pi\Gamma}\frac{\partial\omega}{\partial x}, \tag{40}$$

$$C_3 = \frac{1}{v\pi^3}\frac{\partial\pi}{\partial t}, \tag{41}$$

$$C_4 = \frac{\omega}{v\pi^3}\frac{\partial\pi}{\partial x}, \tag{42}$$

$$C_5 = \frac{\Gamma}{v\pi}, \tag{43}$$

$$C_6 = \frac{1}{v\omega\pi^2}\frac{\partial\omega}{\partial t}, \tag{44}$$

$$C_7 = \frac{1}{v\pi^2}\frac{\partial\omega}{\partial x}, \tag{45}$$

$$C_8 = \frac{g\beta q_{\mathrm{w}}}{k\sqrt{Re_x}}\,\frac{x}{v\omega\pi^2}, \tag{46}$$

$$C_9 = \frac{\omega}{vx\pi^2}. \tag{47}$$

Following (39–47) notations, (36–38) thus reduce to

$$\frac{\mathrm{d}F}{\mathrm{d}\eta} + C_1\,\frac{\mathrm{d}u}{\eta}\,\eta + C_2 u = 0, \tag{48}$$

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\eta^2} - \frac{\mathrm{d}u}{\mathrm{d}\eta}[\eta(C_3 + C_4 u) + C_5 F] - C_6 u - C_7 u^2 + C_8\theta = 0, \tag{49}$$

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}\eta^2} - Pr\left[\frac{\mathrm{d}\theta}{\mathrm{d}\eta}\{\eta(C_3 + C_4 u) + C_5 F\} - C_9 u\theta\right] = 0. \tag{50}$$

Evaluation of the constants: For

$$\frac{C_7}{C_8} = 1 \Rightarrow \omega(x) = \left(\frac{g\beta q_{\mathrm{w}}}{k\sqrt{Re_x}}\right)^{1/2} x. \tag{51}$$

Then from (35), we evaluate u_w

$$u_w(x) = \left(\frac{g\beta q_w}{k\sqrt{Re_x}}\right)^{1/2} x,$$

u_w is found to be a function of x and the fluid parameters. This result is comparable with Kumari et al. [5] assumption $u_w = ax$.

For $C_7 = 1$, we obtain

$$\pi = \frac{1}{v^{1/2}}\left(\frac{g\beta q_w}{k\sqrt{Re_x}}\right)^{1/4}. \tag{52}$$

From (52) in (29), the similarity variable η is explicitly expressed as

$$\eta = y\left(\frac{g\beta q_w}{\gamma^2 k\sqrt{Re_x}}\right)^{1/4} \tag{53}$$

η is found to be a function of y and the fluid conditions. This results is in agreement with Kumari et al. [5] assumption $\eta = (a/\gamma)^{1/2} y$

For $C_5 = 1$ in (43), we obtain

$$\Gamma = v^{1/2}\left(\frac{g\beta q_w}{k\sqrt{Re_x}}\right)^{1/4}. \tag{54}$$

From (54) and (34), the horizontal velocity $v(\eta)$ is derived

$$v(\eta) = v^{1/2}\left(\frac{g\beta q_w}{k\sqrt{Re_x}}\right)^{1/4} \quad F(\eta).$$

Finally, π being constant as it appears in (53), implies

$$C_1 = C_3 = C_4 = 0$$

and as $\omega(x)$ in (51) is not a function of time, then $C_6 = 0$, while replacing for ω, π and Γ in (39), we get $C_2 = 1$, and for ω and Γ in (47), we get $C_9 = 1$.

The previous results are summarized as follows:

$$C_1 = C_3 = C_4 = C_6 = 0, \quad C_2 = C_5 = C_7 = C_8 = C_9 = 1$$

and the system of ordinary differential equations (48)–(50) reduces to

$$\frac{dF}{d\eta} + u = 0, \tag{55}$$

$$\frac{d^2u}{d\eta^2} - F\frac{du}{d\eta} - u^2 + \theta = 0, \tag{56}$$

$$\frac{d^2\theta}{d\eta^2} - Pr\left(\frac{d\theta}{d\eta}F + u\theta\right) = 0, \tag{57}$$

with the related boundary conditions:

$$\begin{aligned}&F(0)=0,\\&u(0)=1,\quad u(\infty)=0,\\&\frac{\mathrm{d}\theta(0)}{\mathrm{d}\eta}=-1,\quad \theta(\infty)=0\end{aligned}\tag{58}$$

We now prove $\mathrm{d}\theta(0)/\mathrm{d}\eta=-1$. Starting with the boundary condition in (8) $\partial\theta(x,0,t)/\partial y=-\sqrt{Re_x}/x$

$$\text{where}\quad Re_x=\frac{u_s x}{\gamma},\tag{59}$$

u_s is the average velocity in the vertical direction "x". In the expression Re_x, let

$$u_s=u_{\mathrm{w}}.\tag{60}$$

Invoking (52), u_{w} may be written as

$$u_{\mathrm{w}}=x\pi^2\gamma.\tag{61}$$

From (61) and (59) we get

$$Re_x=x^2\pi^2.\tag{62}$$

By chain rule, (8) reduces to

$$\frac{\partial\theta(0)}{\partial\eta}\,\frac{\partial\eta}{\partial y}=-\frac{\sqrt{Re_x}}{x}\tag{63}$$

from (29) and (62) in (63) we obtain $\mathrm{d}\theta(0)/\mathrm{d}\eta=-1$#

The system of ordinary differential equations (55)–(57) being a boundary value problem highly non-linear in $u(\eta)$, $F(\eta)$ and $\theta(\eta)$, we solve it numerically using the shooting method.

6. Results and discussion

An observation of the system of ordinary differential equations (55)–(57) show that the only parameter remaining from the transformation of the original flow equations (1–3) is the Prandtl number. We did study the effect of this parameter on the vertical velocity $u(\eta)$ and temperature $\theta(\eta)$ for different values of $Pr=1, 2, 5, 10$. The results obtained were plotted.

- The vertical velocity $u(\eta)$ plotted in Fig. 2 shows a decrease of velocity for larger Prandtl number. In other words, the velocity $u(\eta)$ inside the layer decreases, with the increase of fluid viscosity.

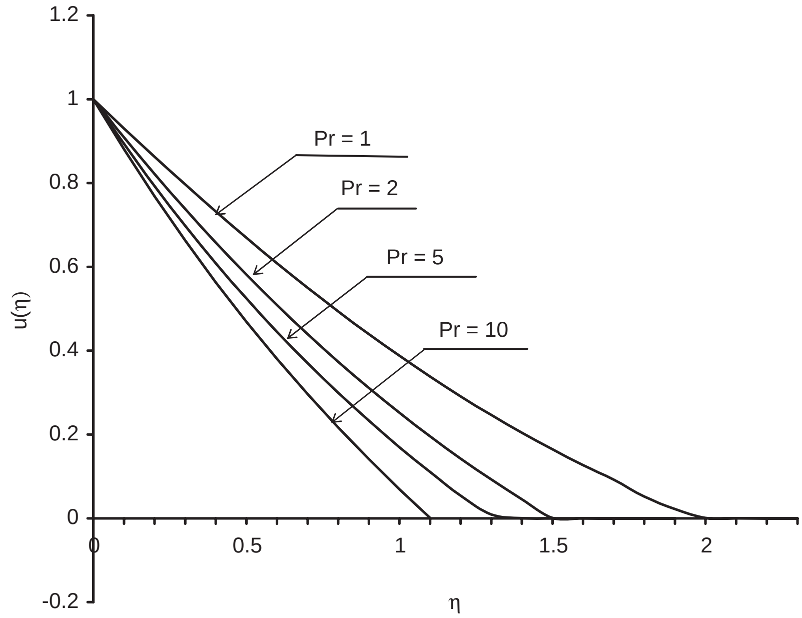

Fig. 2. Variation of vertical velocity "u" of the fluid for different Prandtl numbers.

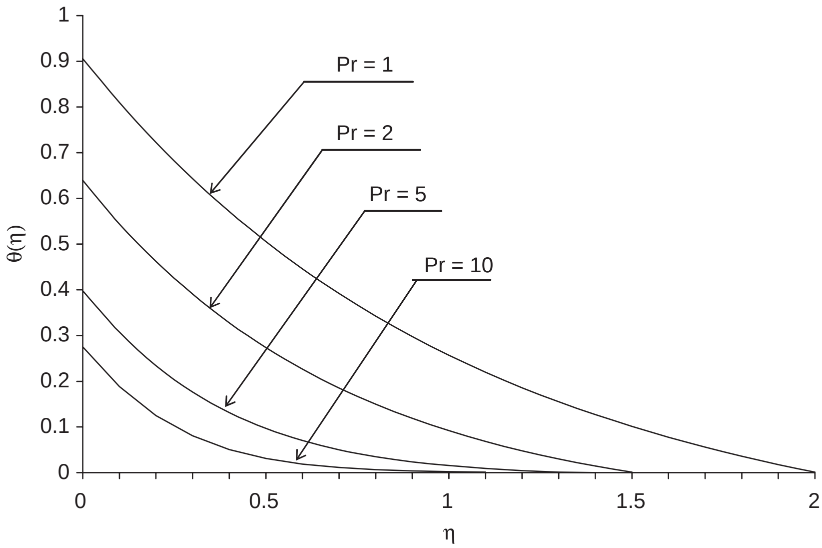

Fig. 3. Variation of fluid temperature θ, for different Prandtl numbers.

- Similarly, the temperature $\theta(\eta)$ is plotted versus η in Fig. 3 for different values of $Pr = 1, 2, 5, 10$. The plot shows a decrease of temperature at the wall temperature $\theta(0)$ for larger Pr. This decrease in $\theta(0)$ is be explained by the fact that for large viscosity heat losses increase, as the boundary layer get thinner.

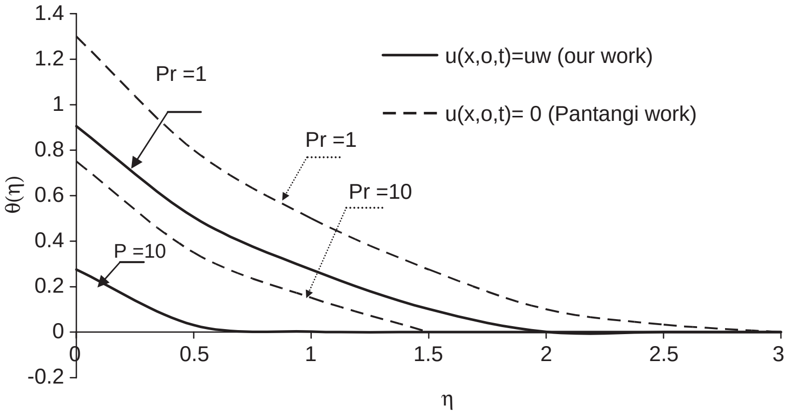

Fig. 4. Comparison of $\theta(\eta)$ with Pantangi et al. [9] where $u(x, o, t) = 0$, while here $u(x, o, t) = u_{\mathrm{W}}$.

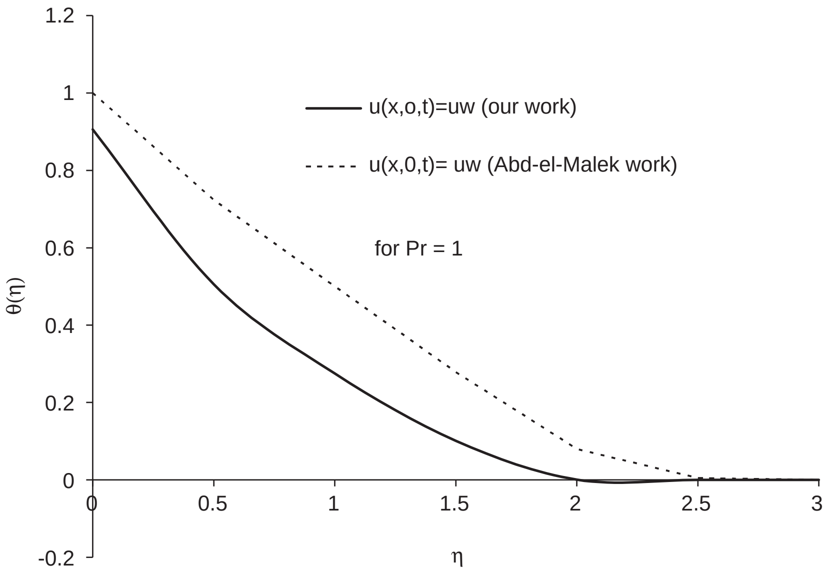

Fig. 5. Comparison of $\theta(\eta)$ profile with Abd-el-Malek [2].

Comparing this work with two previous studies having different boundary conditions, we found the following.

- We compared our results with Pantangi et al. [9] work, for a Newtonian index "$n = 1$". As we had in common a constant heat flux at the wall $\partial\theta(0)/\partial\eta = -1$ while the velocity of the fluid at the wall was different, we only compared the temperature inside the layer for $Pr = 1$ and 10. The comparison illustrated in Fig. 4 shows that $\theta(\eta)$ in our work is always lower than [9] results. This is due to the

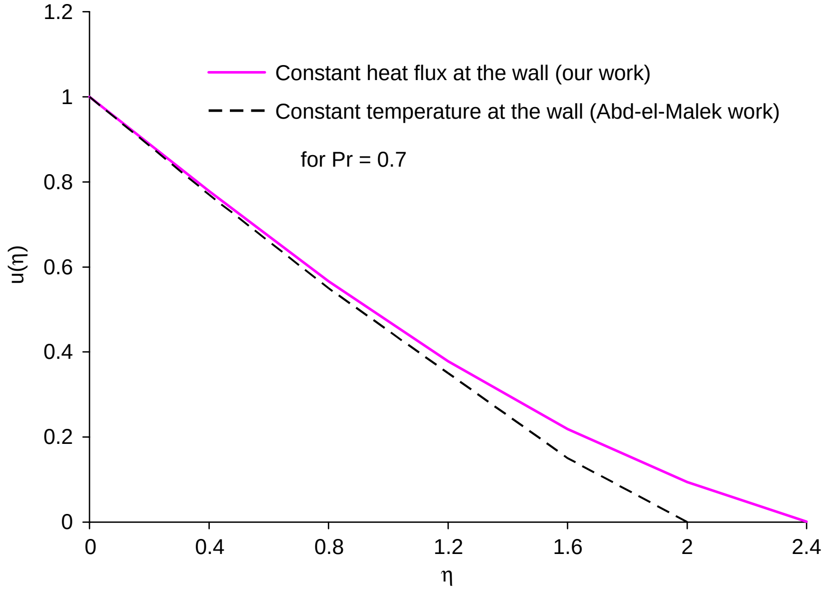

Fig. 6. Comparison of vertical velocity $u(\eta)$ with Abd-el Malek [2].

stagnancy of fluid at the wall $u(0) = 0$ in the case of Pantangi, keeping the temperature higher at the vicinity of the wall.

- We then compared our results with Abd-el-Malek et al. [2], where the temperature of the plate at the wall is constant $\theta(0) = 1$, while in our case the heat flux is constant $\partial\theta/\partial\eta = -1$. The comparison is illustrated in Fig. 5 for $Pr = 1$, where it appears that the temperature $\theta(\eta)$ evaluated here is smaller than [2] results. This is due to the different heating conditions at the wall.

  - Finally, we compared $u(\eta)$ results with Abd-el-Malek [2] for $Pr = 0.7$. This comparison illustrated in Fig. 6, shows a quicker decline of $u(\eta)$ our case then in [2]. Again this difference is due to the different heating conditions at the wall. In [2], the constancy of temperature at the wall causes larger buoyancy.
  - As a conclusion, we might say that the results obtained using the group method accurately describe the free-convective flow induced bya moving heated plate.

Acknowledgements

The author wishes to thank the referee for their remarks, which made him compare the results with previous works.

References

[1] M.B. Abd-el-Malek, N.A. Badran, Group method analysis of unsteady free-convective boundary-layer flow on a non isothermal vertical flat plate, J. Eng. Math. 24 (1990) 343–368.

[2] M. Abd-el-Malek, M. Kassem, M.L. Mekky, Similarity solutions for unsteady free-convection flow from a continuous moving vertical surface, J. Comput. Appl. Math. 164 (2004) 11–24.
[3] Y.Z. Boutros, M.A. Abd-el-Malek, N.A. Badran, Group theoretic approach for solving time-independent free-convective boundary-layer flow on a non isothermal vertical flat plate, Arch. Mech. 42 (1990) 377–395.
[4] R.P. Heinisch, R. Viskanta, R.M. Singer, Approximate solution of the transient free convection laminar boundary layer equations, Math. Phys. 20 (1969) 19–33.
[5] M. Kumari, A. Slaouti, H.S. Takhar, S. Nakamura, G. Nath, Unsteady free convection flow over a continuous moving vertical surface, Acta Mech. 116 (1996) 75–82.
[6] A.J.A. Morgan, The reduction by one of the number of independent variables in some systems of partial differential equations, Quart. J. Math. 3 (1952) 250–259.
[7] M.J. Moran, R.A. Gaggioli, Similarity analysis via group theory, AIAA J. 6 (1968) 2014–2016.
[8] M.J. Moran, R.A. Gaggioli, Reduction of the number of variables in systems of partial differential equations with auxiliary conditions, SIAM J. Appl. Math. 16 (1968) 202–215.
[9] U.S. Pantangi, G.Dr. Ramamurthy, U.M. Vanamala, Heat transfer analysis of a non-Newtonian power law fluid over a vertical plate with constant heat flux, IE(I) J.-CH 84 (2003) 25–32.
[10] J.C. Williams, J.C. Mulligan, T.B. Rhyne, Semi-similar solution for unsteady free-convective boundary layer-flow on a vertical flat plate, J. Fluid Mech. 175 (1987) 309–332.